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ON METRIZATION AND DISCRETE COLLECTIONS OF POINT SETS

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Recently, Bing [2] has formally raised the question as to whether the existence of a nonmetrizable normal Moore space implies that there is a normal Moore space which contains a discrete subset with respect to which the space is not collectionwise normal. Following the convention of Bing, such a space is called a Counterexample of Type D. Already known is that the existence of a normal, locally compact, nonmetrizable Moore space implies the existence of a Counterexample of Type D [12, Theorem 4], and that the existence of a normal, separable, nonmetrizable Moore space give the same implication [2, Theorem 3].

The primary purpose of this paper is to prove that there is a Counterexample of Type D provided that there exists a normal, locally separable, nonmetrizable Moore space. The results of Theorem 1 are similar to those established by Grace [5], except that the included results are cast in the setting of a Moore space and offer hypotheses which are stated in terms of discrete collections of point sets.

A Moore space is one which satisfies the first three parts of Axiom 1 of [11]. For other definitions and results related to the question of metrization of normal Moore spaces, refer to [1], [3], [4], [6], [7], [8], [9], [10], [13], [14], and [15].

Theorem 1. Suppose that  $S$  is a Moore space and there exists an open covering  $H$  of  $S$  such that if  $G$  is a discrete collection of point sets refining  $H$ , then the boundary of  $G^*$  is (strongly) screenable. Then  $S$  is (strongly) screenable.

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Proof. Suppose that  $H$  is an open covering of  $S$  and, by [1, Theorem 9],  $H_1, H_2, H_3, \dots$  is a sequence of discrete collections of closed sets such that, for each  $i$ ,  $H_i$  refines  $H$ ,  $H_{i+1}^*$  contains  $H_i^*$ , and  $\bigcup H_i^* = S$ . For each  $i$  and each element  $h$  of  $H_i$ , let  $g_h$  denote the interior of  $h$  if  $h$  contains an open subset. For each  $i$ , denote by  $V_i$  the collection to which  $v$  belongs if and only if there exists an element  $h$  of  $H_i$  such that  $v = g_h$ . Then  $V_i$  is a discrete collection of open sets. Denote by  $V_i'$  the collection to which the set  $v$  belongs if and only if there exists an element  $h$  of  $H_i$  such that  $v$  is the boundary of  $h$ . It follows that  $V_i'$  is a discrete collection of closed sets,  $V_i'^*$  is the boundary of  $H_i^*$ , and  $V_i^* + V_i'^*$  contains  $H_i^*$ . From the hypothesis of the theorem, it follows that  $V_i'^*$  is (strongly) screenable. Thus, there exists a sequence  $U_{i_1}, U_{i_2}, \dots$  of (discrete) collections of mutually exclusive open sets in  $S$  such that  $V_i'^*$  is covered by  $\bigcup_j U_{i_j}$  and each  $U_{i_j}$  is a refinement of  $H$ . The sequences  $\{V_i\}_{i=1}^\infty$  and  $\{U_{i_j}\}_{i=1, j=1}^\infty$  give rise to a sequence of collections of open sets satisfying the definition of (strong) screenability.

Theorem 2. Suppose that  $S$  is a Moore space which is locally separable, normal, and nonmetrizable. Then  $S$  is a Counterexample of Type D.

Proof. Denote by  $H$  an open covering of  $S$  such that each element of  $H$  is separable and by  $G$  a discrete collection of point sets such that  $G$  refines  $H$ . If  $S$  is not a Counterexample of Type D, then  $S$  is collectionwise normal with respect to each uncountable discrete set. It follows that if  $g$  is an element of  $G$  and  $M$  is an uncountable subset of  $\overline{g}$ , then  $M$  is not discrete since  $g$  is a subset of a separable element of  $H$ . Then each uncountable

subset of  $\overline{g}$  has a limit point and by [11, Page 9, Theorem 22],  $\overline{g}$  is completely separable.

Since  $G$  is discrete, the boundary of  $G^*$  is the sum of the boundaries of the elements of  $G$ . If  $g$  is an element of  $G$ , let  $B(g)$  denote the boundary of  $g$ . To prove that the boundary of  $G^*$  is screenable, let  $V$  denote a collection of open sets covering  $S$  and denote by  $U$  a collection of open sets covering  $S$  such that no element of  $U$  intersects two elements of  $G$ . Since  $B(g)$  is completely separable, there exists a collection  $H_g$  of open sets such that  $H_g$  is countable,  $H_g$  covers  $B(g)$ , and if  $P$  is a point of  $B(g)$  and  $O$  is an open set containing  $P$ , then some element of  $H_g$  contains  $P$  and is a subset of  $O$ . Now denote by  $M_1$  a point set such that, for each element  $g$  of  $G$ ,  $M_1$  contains exactly one point of  $B(g)$  and  $M_1$  is a subset of  $\bigcup_{g \in G} B(g)$ . It follows that  $M_1$  is a discrete point set and, since  $S$  is collectionwise normal with respect to each such set, there is a collection  $W_{M_1}$  of mutually exclusive domains such that  $W_{M_1}$  refines  $U$  and  $W_{M_1}$  covers  $M_1$ . If  $P$  is a point of  $M_1 \cdot B(g)$  for some  $g$ , then some element  $h_p$  of  $H_g$  contains  $P$  and is a subset of that element of  $W_{M_1}$  which contains  $P$ . Denote by  $V_1$  the collection to which  $v$  belongs if and only if there is a point  $P$  of  $M_1 \cdot B(g)$  such that  $v$  is  $h_p$ . Next, let  $M_2$  denote a point set such that, for each element  $g$  of  $G$ ,  $M_2$  contains exactly one point of  $B(g) - V_1^*$  (if that set exists) and  $M_2$  is a subset of  $\bigcup_{g \in G} B(g)$ . As with  $M_1$ , it follows that  $M_2$  is a discrete set and that there exists a collection  $V_2$  of mutually exclusive open sets such that  $V_2$  covers  $M_2$ , each element of  $V_2$  is an element of  $H_g$  for some  $g$  of  $G$ , and no element of  $V_1$  is an element of  $V_2$ . This process continued indefinitely gives rise to a countable sequence (not necessarily simply infinite, though the construction could have been defined to give such a sequence)  $V_1, V_2, \dots, V_\alpha, \dots$  such

that each  $V_\alpha$  is a collection of mutually exclusive domains which refines  $U$  and it is clear that the sum of the elements of the collections of the sequence covers the boundary of  $G^*$ . That the sequence is only countable follows quickly from the fact that  $H_g$  is countable for each  $g$ .

This has established that the boundary of  $G^*$  is screenable and, by Theorem 1, and [1, Theorem 8],  $S$  must be metrizable and a contradiction is reached to the assumption that  $S$  is not a Counterexample of Type D.

The statement that  $S$  satisfies a Souslin property locally means that if  $P$  is a point of  $S$  then there exists an open set  $O$  containing  $P$  such that  $O$  does not contain uncountably many mutually exclusive domains.

Theorem 3. If  $S$  is a normal, nonmetrizable Moore space which satisfies a Souslin property locally, then  $S$  is a Counterexample of Type D.

Proof. Consider an open covering of  $S$  such that each element of that covering does not contain uncountably many mutually exclusive domains and let  $G$  be any discrete collection of point sets refining that open cover. It follows as in Theorem 2 that the closure of each element of  $G$  must be completely separable and this allows application of Theorem 1 to complete the argument as in the preceding theorem.

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